The Minimum Correlation Algorithm:

A Practical Diversification Tool

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Discussion and Results

> *"But divide your investments among many places,*
> *for you do not know what risks might lie ahead."*
> *Ecclesiastes*

Diversification is possibly the most widely accepted principle in finance. The roots of this concept go as far back to ancient times where individuals were advised to keep an equal proportion of their wealth in real estate (land), equities (business), and cash (liquid holdings). However, the modern financial interpretation of diversification is not well-understood despite its acceptance. Markowitz was the first person to numerically quantify the impact of diversification by using the inputs of returns, correlations and volatility in a quadratic programming framework. In the Modern Portfolio Theory framework, returns are the simple average of past returns, correlations represent the standardized measure of the strength and slope of the relationship between past returns, and volatility is conceptually the average squared difference of returns from their simple average. This approach highlighted two counter-intuitive findings: 1) the portfolio volatility of say two assets is always lower than the weighted average volatility of each asset as long as the correlation is less than one. 2) the relationship between correlation and the portfolio risk/average risk (diversification ratio) is curvilinear. In fact, the risk reduction formula is described by several squared components. This is why "quadratic" programming is required to solve portfolio problems- the term quadratic is derived from the Latin term "quadratus" which is the

Latin word for “square.”The calculation for the volatility of a two asset portfolio (from Wikipedia: http://en.wikipedia.org/wiki/Modern_portfolio_theory) for example is:

$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_A \sigma_B \rho_{AB}$$

where the final term represents the adjustment for the correlation between assets A and B. But the elegance of the formula hides much of the ugly truth. The concept of diversification is more nuanced and complex than most people realize. The global financial crisis today has served to highlight serious deficiencies in our true understanding of the subject. Multi-asset and large-scale risk allocation is a highly dimensional problem for which there is no closed-form solution. The integration of both qualitative and quantitative considerations are necessary in the context of an adequate solution. As with most cases in the investment world, the approach is always polarized towards philosophical preference: Discretionary portfolio managers realize the limitations of basic numerical solutions and how to apply “common sense” to capture these deficiencies. The drawback to their approach is that they tend to substitute rules of thumb that lack rigor, accuracy, or sufficient complexity. In contrast, quantitative portfolio managers often fail to account for both common sense and the violation in the assumptions underlying their models that are present in real-life problems. To their credit, they use models that are far more capable of handling such complex problems in an unbiased manner.

Diversification is more than just a prescription for holding a large number of assets, it is also about the relationships between those assets, the relative risk contribution of those assets, and also possibility of a sudden large loss or default. There is a much simpler formula that is straightforward to derive from portfolio mathematics that neatly captures many of the nuances of diversification and its contribution to risk reduction in a portfolio context:

Portfolio Variance= K x Average Asset Variance +(1-K) x Average Asset Covariance

where K= 1/number of assets in the portfolio

If we translate this formula by substituting standard deviation for the variance and correlation for the covariance (see http://cssanalytics.wordpress.com/2012/07/16/diversification-and-risk-reduction/ for the derivation) it becomes clear that portfolio risk is a function of 1) the average asset risk 2) the constant “K” and 3) the average correlation between assets. The constant “K” can be thought of as an inverse multiplier(1/number of assets) that weights the contribution of average asset risk to portfolio risk as a function of the number of assets in the portfolio. The higher “K” is, the less important the average asset risk is and the more important the average correlation between assets becomes. Other things being equal, the more assets in the portfolio at a given level of average correlation will reduce risk at the portfolio level. The constant “K” becomes quite small with even 10 assets, making the contribution of the first term of the equation less important. In fact for large data sets with greater than 50 assets, the portfolio risk formula can actually be roughly approximated by :

Portfolio Risk (for large datasets)= Average Asset Correlation x Average Asset Risk

recall that Risk Reduction due to diversification= Portfolio Risk/Average Asset Risk-1

by substituting in our approximation for portfolio risk we are left with the following simplification:

Risk Reduction (due to diversification)= **Average Asset Correlation -1**

This makes intuitive sense- to reduce portfolio risk, we generally need to find assets that have a low average correlation. The average asset correlation is the simple average of the correlations within the matrix.

The table below shows the relationship between the number of assets held and the average correlation. Here we assume that each asset has the same level of risk—a standard deviation of 10%. As you can clearly see the major driver of risk reduction is the average correlation. In contrast the number of assets held has a rapidly diminishing marginal impact on portfolio risk.

	Average Correlation Between Assets								
Number of Assets	**1**	**0.75**	**0.5**	**0.25**	**0**	**-0.25**	**-0.5**	**-0.75**	**-1**
2	10%	9.35%	8.66%	7.91%	7.07%	6.12%	5%	3.54%	0.00%
5	10%	8.94%	7.75%	6.32%	4.47%	0%	0%	0%	0%
10	10%	8.80%	7.42%	5.70%	3.16%	0%	0%	0%	0%
20	10%	8.73%	7.25%	5.36%	2.24%	0%	0%	0%	0%
50	10%	8.69%	7.14%	5.15%	1.41%	0%	0%	0%	0%
100	10%	8.67%	7.11%	5.07%	1%	0%	0%	0%	0%
500	10%	8.66%	7.08%	5.01%	0.45%	0%	0%	0%	0%
1000	10%	8.66%	7.07%	5.01%	0.32%	0%	0%	0%	0%

Diversification Ratio

In the table the effect of correlations on diversification is easy to see because the risk is standardized to 10% across assets. The risk reduction can be observed by comparing the portfolio risk as a fraction of 10%. So a portfolio that has a 6% standard deviation implies that risk was reduced by 40% through diversification (1-6%/10%). In practice, assets have variable standard deviations and it is more difficult to capture the true impact of correlations on risk reduction. The Diversification Ratio as a means of capturing the relationship between the actual portfolio standard deviation and the weighted average of the individual standard deviations for each of the assets in the portfolio. More formally this can be computed as:

Diversification Ratio (D)= 1- Portfolio Risk(Standard Deviation)/Weighted Average Asset Risk

The higher the fraction is, the more confident we can be that we have minimized average portfolio correlation. Alternatively, this can be viewed as representing the degree of efficiency in using only the correlation matrix data for a given set of assets. [Toward Maximum Diversification by Y. Choueifaty, Y. Coignard]

Risk Dispersion

Another important component of diversification is distributing risk over as many assets as possible. This is perhaps the most intuitive aspect of diversification, especially when considering individual stocks or bonds. Common practice is to hold a large number of different securities in equal weight to avoid "concentration risk." Large weightings in individual securities in relation to an equal weighting are considered to increase concentration risk—or the possibility that an unlucky event can significantly hurt portfolio performance. Despite the common use of this heuristic, the actual underlying goal is to balance risk contributions. In other words, you want to ensure that the contribution of each security or asset to total portfolio risk is fairly equal. (a good overview of the concept of risk contributions can be found here: http://cssanalytics.wordpress.com/2012/07/10/risk-decomposed-marginal-versus-risk-contributions/)

In this paper we use the term "risk dispersion" to measure the degree to which the contribution of risk is dispersed evenly across assets. To measure this equality, we use the Gini coefficient(a good overview of the

Gini coefficient can be found here: http://cssanalytics.wordpress.com/2012/02/24/i-dream-of-gini/)The theory is that the more equal the risk dispersion the less likely that portfolio risk will be impacted by any one asset in the portfolio. For example, we would want a portfolio to be able to withstand holding a position in Enron, a large earnings miss by a glamor stock, large and unexpected government intervention, or possibly even a bond default. In gambling terms, we would like to strive to make as many equally sized "bets" as possible by normalizing the size of each bet and also considering the degree of correlation between bets.

As a consequence, good risk dispersion also ensures a more balanced contribution of asset returns to the portfolio. Consider that in blackjack the payoff to winning is always symmetrical with the dollar amount that is bet- that is the payoff odds are fixed. In the financial markets, the payoffs are variable and unknown. It is important to consider that two highly correlated assets can have substantially different absolute and risk-adjusted returns. Consider the more familiar example of a stock like Apple Computer, that can have a high correlation to the S&P500 and similar risk but substantially greater performance. Since we are striving to create a "forecast-free" algorithm (see http://cssanalytics.wordpress.com/2011/08/09/forecast-free-algorithms-a-new-benchmark-for-tactical-strategies/) we want to allocate effectively to capture returns under the conditions of uncertainty . Risk dispersion is a method of minimizing the risk of regret to holding a portfolio that might underperform a portfolio with a similar composition. The more equal the risk dispersion, the less likely we will regret the decision to hold a particular portfolio.

More formally, the equation for risk dispersion is:

Risk Dispersion (RD)= 1- Gini Coefficient (of risk contributions across assets)

Composite Diversification Indicator (CDI)

We have described the two major elements of diversification, and it makes sense to combine them together into one number. The details of the formula can be found in the appendices, but generally speaking the CDI is:

CDI= w*D+(1-w)*RD

The weight (w) can be pre-specified as .5- or just the simple average between the two metrics. Or the weight can be varied in some form of optimization. In trying to capture the "Diversification Efficiency", we looked at the average rank of the CDI for a given algorithm versus competing algorithms across all weighting schemes. In this way, we do not have a pre-specified/hindsight bias. Furthermore we are much more invariant to discontinuities in the scale of the CDI by weight bias. By combining this average rank for CDI by algorithm across multiple datasets we can get a better sense of the “Diversification Efficiency.”

Minimum Correlation Algorithm (MCA)

Earlier we demonstrated the importance of minimizing the average correlations to reduce variance. Since the average correlation is a function of the weighting on the correlation matrix, and each column of the matrix can represent the weight of a given asset, if we want to minimize average correlations we should weight the assets that have the lowest average correlation to all other assets (by column) more favorably than other assets. This shifts the average to be weighted by those assets that are “diversifiers”, or tend to have low correlations to the rest of the portfolio.

The proportional weighting concept is used in favor of precise optimization to 1) enhance the speed of calculation- especially across large data sets 2) simplicity of calculation- most optimization procedures are either difficult to understand for practitioners or difficult to implement (or both) and 3) most importantly to ensure a robust solution given the noise embedded in time series data. In information theory there is the concept of a center of gravity or proportional weighting that maximizes the desired output while minimizing the chance of regret. The log optimal portfolio- or the portfolio that maximizes long-term returns- is calculated as a proportional investment in each asset based on their geometric return. A different representation is found in Cover’s Universal Portfolio, where all constantly rebalanced portfolios are weighted in proportion to their compound return. In general, proportional algorithms are: 1) simple 2) fast and 3) maximize or minimize a target while doing a good job of minimizing regret. [AKSUYEK, INFORMATION THEORY AND PORTFOLIO MANAGEMENT, 2008.]

We can therefore proportionately weight the average correlations by asset, which should approximate the effect of minimizing correlations without having to rely on a precise solution using either matrix inversion or a solver. Furthermore, it logical to find a long-only solution- which is theoretically more stable and also behaves as a form of shrinkage. To accomplish these goals, it is necessary to convert the correlation matrix to either a relative scale that does not have negative values, and/or convert the average correlations to ensure that it does not have negative values. Furthermore, the weighting of individual correlations must be done in a stable manner which is best accomplished using a rank-weighted function. Finally, we need to adjust/normalize each asset for volatility to ensure that they have equivalent risk. This process is accomplished differently with two different variations of the minimum correlation algorithm (details and example calculations are in the appendices):

Mincorr: this method normalizes both the correlation matrix, and the rank-weighted average normalized correlations and proportionately weights (re-leveraging to 1) by the lowest average correlations. This first step forms the base weight. A risk-parity multiplier is applied- each is asset is sized according to inverse of their individual standard deviations. The final allocation is derived by re-leveraging these fractions to sum to 1.

Mincorr2: this method takes the raw correlation matrix, and then finds the rank-weighted average normalized correlations and proportionately weights (re-leveraging to 1) by the lowest average correlations. This first step forms the base weight. A risk-parity multiplier is applied- each is asset is sized according to inverse of their individual standard deviations. The final allocation is derived by re-leveraging these fractions to sum to 1.

The desirable feature of a heuristic solution is that by averaging correlations we achieve a few desirable benefits: 1) less sensitivity to estimation error: a given correlation coefficient between any two assets is less important since we are measuring the average correlation for each asset versus the universe. The estimation error is averaged in this process, which creates a simple ensemble of predictors (sample correlation coefficients) that reduce the possibility of extreme errors. If the sample correlation is a poor forecast for a particular pair of assets, then it will exacerbate errors in the final portfolio using traditional optimization methods (matrix inversion, and solver/brute-force approaches). 2) all assets are included in the final weighting scheme. This promotes the concept of risk dispersion,and reduces the risk of large unexpected tail events for individual assets and the risk of holding assets that underperform the universe.

Methodology

We used four different datasets for testing detailed in the appendices: 1) a list of Dow stocks with a long history 2) a list of 8 highly liquid and diverse asset class ETFs 3) a dataset including 50 futures and forex markets 4) the current Nasdaq 100. All calculations and analysis were handled using "R" software. We compared the two MCA variants: mincorr/MCA and mincorr2/MCA2, versus equal weight, risk parity, minimum variance (minvar), and maximum diversification (maxdiv). All algorithms were computed with long-only constraints, and in this case for simplicity the leverage was re-scaled to 100%. We used a 60-day look-back for correlations and volatility using simple in-sample (historical window) estimation. Rebalancing for all datasets was done weekly with the exception of the futures dataset which was done monthly.

Results

The tests show that the MCA algorithms are highly competitive across all datasets in terms of risk-adjusted returns. We compared the average rank across data sets of each algorithm versus the others in terms of Sharpe ratio. (more detailed results and a complete array of metrics are available in the appendices) The results are summarized in the table below in order of best to worst:

	Sharpe Rank
MCA	1.75
Minvar	2.25
MCA2	3
Maxdiv	4.25
Risk Parity	4.25
EqualWeight	5.5

Interestingly, despite the use of a heuristic method, mincorr produced the best combination of results across datasets in terms of Sharpe ratio. Minvar had the second best risk-adjusted returns on average with Mincorr2 following just in behind. Risk Parity and maxdiv were tied, with maxdiv showing the largest edge in the ETF asset class dataset while risk parity performed better on average over the other datasets. Equal weight was the worst performer in terms of risk-adjusted returns which is to be expected since it does not adjust for either correlations or volatility.

The most important finding was not the risk-adjusted performance, but rather the "Diversification Efficiency." In this case, both the MCA algorithms outperformed their peers substantially. The table below summarizes the average rank from best to worst across datasets using the CDI (with the average of the CDI ranks across weight schemes between D and RD).

Diversification Efficiency

MCA2	1.5
MCA	1.75
Risk Parity	2.75
Maxdiv	4.5
Equal Weight	4.75
Minvar	5.75

In this comparison, MCA2 is the algorithm that has the best diversification efficiency which makes sense given that it is less biased towards anti-correlated assets than mincorr. Both MCA variants showed far superior diversification efficiency versus other algorithms. Risk Parity was the next best, and suffered from the fact that the correlation matrix is not considered in portfolio formation. In metric terms, the diversification ratio was poorly maximized. However, Risk Parity did a good job in terms of maximizing risk dispersion. Maxdiv followed behind Risk Parity, but suffered from poor risk dispersion. In contrast, it did a very good job of maximizing the diversification ratio. Equal weight was surprisingly better than minimum variance, having done a much better job at risk dispersion despite doing a poor job at maximizing the diversification ratio. Finally, minvar did the worst job of all the algorithms in terms of diversification efficiency. This is not surprising given excessive concentration on the least volatile assets with low correlations. In other words, minvar did a poor job of risk dispersion.

Conclusion

In this paper, we introduced two heuristic algorithms for effective portfolio diversification and passive investment management. The concept of proportional allocation was introduced as being a useful concept for creating algorithms that are fast, simple, and robust. We also introduced the Composite Diversification Indicator and a useful test for Diversification Efficiency. While the testing performed in this article was not exhaustive across datasets and parameters, a reasonable analysis shows that both MCA variants are very competitive on a risk-adjusted basis to other algorithms. More importantly the MCA variants did the best in terms of diversification efficiency versus all other algorithms. MCA has been demonstrated to be very practical and useful allocation tool for portfolio management that is accessibly and easy to understand. It is an excellent alternative to Risk Parity, Minimum Variance and Maximum Diversification.

Appendix: Correlation and Covariance Matrix calculations

The Pearson correlation is +1 in the case of a perfect positive (increasing) linear relationship (correlation), −1 in the case of a perfect decreasing (negative) linear relationship, and some value between −1 and 1 in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero there is less of a relationship. The closer the coefficient is to either −1 or 1, the stronger the correlation between the variables.

The Pearson correlation between two variables X and Y with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as:

$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\mathrm{E}[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

where E is the expected value operator and covariance $\Sigma_{X,Y} = \operatorname{cov}(X,Y) = \mathrm{E}[(X-\mu_X)(Y-\mu_Y)]$

The Covariance Matrix for the random variables $\mathrm{X}_1 \ldots \mathrm{X}_N$ is:

$$\Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \cdots & \Sigma_{1,N} \\ \Sigma_{2,1} & \Sigma_{2,2} & \cdots & \Sigma_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{N,1} & \Sigma_{N,2} & \cdots & \Sigma_{N,N} \end{bmatrix}$$

The Correlation Matrix for the random variables $\mathrm{X}_1 \ldots \mathrm{X}_N$ is:

$$\rho = \begin{bmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,N} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N,1} & \rho_{N,2} & \cdots & \rho_{N,N} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,N} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N,1} & \rho_{N,2} & \cdots & 1 \end{bmatrix}$$

Steps to compute Covariance Matrix and Correlation Matrix from the raw data:
Suppose $\mathrm{X} = [\mathrm{X}_1 \ldots \mathrm{X}_N]$ is matrix of N variables, and each variable $\mathrm{X}_i = [x_{i,1} \quad x_{i,2} \quad \ldots \quad x_{i,T}]$ has T observations

- transform X into matrix of deviations $(\mathrm{X}_i - \mu_{\mathrm{X}_i})$

  $D = \mathrm{X} - \frac{11^{/}\mathrm{X}}{T}$, where 1 is an T x 1 column vector of ones

- the Covariance Matrix $\Sigma = \frac{D^{/}D}{T}$
- the Correlation Matrix $\rho = \frac{\Sigma}{\sigma^{/}\sigma}$, where σ is an N x 1 column vector of standard deviations.

  $\sigma = sqrt[diag(\Sigma)]$

The above material is based on the following references:
http://en.wikipedia.org/wiki/Correlation_and_dependence
http://en.wikipedia.org/wiki/Covariance_matrix
http://stattrek.com/matrix-algebra/covariance-matrix.aspx
http://www.riskglossary.com/link/correlation.htm

Sample R code to compute Covariance Matrix and Correlation Matrix from the raw data:

```
N=10
T=100

X = matrix( rnorm(N*T), nr=T, nc=N )
ones = rep(1,T)

D = X - (ones %*% t(ones) %*% X) / T
Sigma = (t(D) %*% D) / (T-1)
range(cov(X) - Sigma)

sigma = sqrt(diag(Sigma))
Rho = Sigma / (sigma %*% t(sigma))

range(cor(X) - Rho)
```

Sample R code to compute portfolio risk as a function of Number of Assets and Average Correlation Between Assets:

```
# number of assets
n = 2

# average asset correlation
rho = -0.25

# risk
risk = rep(10/100, n)

# correlation matrix
cor = matrix(rho, n, n)
    diag(cor) = 1

# covariance matrix
cov = cor * (risk %*% t(risk))

# equla weight portfolio
w = rep(1/n, n)

# portfolio risk
sqrt(t(w) %*% cov %*% w)
```

Appendix: Minimum Variance, Maximum Diversification, and Minimum Correlation Portfolios

The Minimum Variance Portfolio is the solution to the following quadratic optimization problem:

$$\begin{array}{ll} \min\limits_{w} & w^{/}\Sigma w \\ \text{subject to} & \sum_{i} w_i = 1 \\ & \forall i : 0 \le w_i \le 1 \end{array}$$

where w is a vector of portfolio weights, Σ is a Covariance Matrix, $sqrt\left(w^{/}\Sigma w\right)$ is portfolio risk. The $\sum_{i} w_i = 1$ constraint ensures portfolio is fully invested and $\forall i : 0 \le w_i \le 1$ boundaries endure no short selling.

We used "quadprog" library to solve quadratic optimization problem. The "quadprog" package implements the dual method of Goldfarb and Idnani (1982, 1983) for solving quadratic programming problems.

The Maximum Diversification Portfolio is the solution to the following quadratic optimization problem:

Step 1: $$\begin{array}{ll} \min\limits_{w} & w^{/}\rho w \\ \text{subject to} & \sum_{i} w_i = 1 \\ & \forall i : 0 \le w_i \le 1 \end{array}$$

Step 2: $$w_i^* = \frac{w_i / \sigma_i}{\sum_{i} w_i / \sigma_i}$$

where w^* is a vector of portfolio weights, ρ is a Correlation Matrix, σ is a vector of standard deviations. In the Step 1, we find portfolio weights assuming all assets have the same standard deviation equal to one. In the Step 2, we scale portfolio weights by assets standard deviations and normalize portfolio weights to sum to one.

Toward Maximum Diversification by Y. Choueifaty, Y. Coignard
The Journal of Portfolio Management, Fall 2008, Vol. 35, No. 1: pp. 40-51

The Minimum Correlation Portfolio algorithm is a heuristic method discovered by David Varadi. Below are the steps to compute weights for the Minimum Correlation Portfolio:

1. Compute Pearson Correlation Matrix, ρ
2. Compute mean, μ_{ρ}, and standard deviation, σ_{ρ}, of all elements of the Correlation Matrix, ρ
3. Create Adjusted Correlation Matrix, ρ_A, by transforming each elements of the Correlation Matrix, ρ, from -1 to +1 space to the 1 to 0 space. Map -1 correlation to numbers close to 1 and +1 correlation to numbers close 0. I.e. penalize high correlation and reward low correlation. The mapping is done using 1 – Normal Inverse transformation. I.e. Excel formula =1-NORMDIST($\rho_{i,j}$, μ_{ρ} , σ_{ρ} ,TRUE)
4. Compute average value for each row of the Adjusted Correlation Matrix, ρ_A. These are the initial portfolio weight estimates after transformation, w_T
5. Compute rank portfolio weight estimates: $w_{RANK} = \dfrac{RANK(w_T)}{\sum_i RANK(w_T)}$
6. Combine rank portfolio weight estimates with Adjusted Correlation Matrix, ρ_A, by multiplying w_{RANK} and ρ_A, and normalizing. $w = \dfrac{w_{RANK} * \rho_A}{\sum_{i} w_{RANK} * \rho_A}$

7. Scale portfolio weights by assets standard deviations and normalize portfolio weights to sum to one
$$w_i^* = \frac{w_i / \sigma_i}{\sum_i w_i / \sigma_i}$$

The Minimum Correlation 2 Portfolio algorithm is a heuristic method discovered by David Varadi. Below are the steps to compute weights for the Minimum Correlation Portfolio:

1. Compute Pearson Correlation Matrix, ρ
2. Compute average value for each row of the Pearson Correlation Matrix, ρ. These are the initial portfolio weight estimates, w_0
3. Compute mean, μ_0, and standard deviation, σ_0, of the initial portfolio weight estimates, w_0
4. Transform the initial portfolio weight estimates, w_0, from -1 to +1 space to the 1 to 0 space. Map -1 correlation to numbers close to 1 and +1 correlation to numbers close 0. I.e. penalize high correlation and reward low correlation. The mapping is done using 1 – Normal Inverse transformation. I.e. Excel formula =1-NORMDIST(w_0, μ_0, σ_0 ,TRUE). These are the initial portfolio weight estimates after transformation, w_T
5. Compute rank portfolio weight estimates: $w_{RANK} = \frac{RANK(w_T)}{\sum_i RANK(w_T)}$
6. Combine rank portfolio weight estimates with Pearson Correlation Matrix, ρ, by multiplying w_{RANK} and $(1-\rho)$, and normalizing. $w = \frac{w_{RANK} * \rho_A}{\sum_i w_{RANK} * \rho_A}$
7. Scale portfolio weights by assets standard deviations and normalize portfolio weights to sum to one
$$w_i^* = \frac{w_i / \sigma_i}{\sum_i w_i / \sigma_i}$$

Appendix: Numerical Example for the Minimum Variance, Maximum Diversification, and Minimum Correlation Portfolios

Let's apply the Minimum Variance, Maximum Diversification, and Minimum Correlation algorithms to the following 3 assets.

The Correlation Matrix, ρ, and vector of standard deviations, σ.

$$\rho = \begin{bmatrix} 1 & 0.90 & 0.85 \\ 0.90 & 1 & 0.70 \\ 0.85 & 0.70 & 1 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 14 & 18 & 22 \end{bmatrix}$$

The Minimum Variance Portfolio is the solution to the following quadratic optimization problem:

$$\begin{array}{ll} \min\limits_{w} & w^{/}\Sigma w \\ subject\ to & \sum\limits_{i} w_i = 1 \\ & \forall i : 0 \le w_i \le 1 \end{array}$$

The solution, $w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, portfolio risk $\sigma_p = 14$

The Maximum Diversification Portfolio is the solution to the following quadratic optimization problem:

$$\text{Step 1: } \begin{array}{ll} \min\limits_{w} & w^{/}\rho w \\ subject\ to & \sum\limits_{i} w_i = 1 \\ & \forall i : 0 \le w_i \le 1 \end{array}$$

the optimal weights, $w = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}$

Step 2: $w_i^* = \dfrac{w_i / \sigma_i}{\sum_i w_i / \sigma_i}$

The final solution, $w^* = \begin{bmatrix} 0 & 0.55 & 0.45 \end{bmatrix}$, portfolio risk $\sigma_p = \sqrt{w^{*/} * (\rho * \sigma * \sigma^{/}) * w^*} = 18.25$

The Minimum Correlation Portfolio algorithm steps:

Step 2: $\mu_\rho = 81.66 = \dfrac{90+85+70}{3}$, $\sigma_\rho = 10.4$

Step 3: ρ_A = 1-NORMDIST($\rho_{i,j}, \mu_\rho, \sigma_\rho$,TRUE) = $\begin{bmatrix} & 21 & 37 \\ 21 & & 87 \\ 37 & 87 & \end{bmatrix}$

Step 4: $w_T = \begin{bmatrix} 0.29 & 0.54 & 0.62 \end{bmatrix}$ => $RANK(w_T) = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$

Step 5: $w_{RANK} = \dfrac{RANK(w_T)}{\sum_i RANK(w_T)} = \begin{bmatrix} 0.5 & 0.33 & 0.17 \end{bmatrix}$

Step 6: $w_{RANK} * \rho_A = \begin{bmatrix} 0.13 & 0.25 & 0.47 \end{bmatrix}$, $w = \dfrac{w_{RANK} * \rho_A}{\sum_i w_{RANK} * \rho_A} = \begin{bmatrix} 0.16 & 0.29 & 0.55 \end{bmatrix}$

Step 7: $w_i^* = \frac{w_i / \sigma_i}{\sum_i w_i / \sigma_i} = [0.21 \quad 0.31 \quad 0.48]$

The final solution, $w^* = [0.21 \quad 0.31 \quad 0.48]$, portfolio risk $\sigma_p = \sqrt{w^{*/} * (\rho * \sigma * \sigma^/) * w^*} = 17.78$

The Minimum Correlation 2 Portfolio algorithm steps:
Step 2: $w_0 = [0.58 \quad 0.53 \quad 0.51]$
Step 3: $\mu_0 = 54.44$, $\sigma_0 = 3.47$
Step 4: w_T = 1-NORMDIST(w_0, μ_0, σ_0 ,TRUE) = $[0.13 \quad 0.63 \quad 0.79]$ => $RANK(w_T) = [3 \quad 2 \quad 1]$
Step 5: $w_{RANK} = \frac{RANK(w_T)}{\sum_i RANK(w_T)} = [0.5 \quad 0.33 \quad 0.17]$
Step 6: $w_{RANK} * (1-\rho) = [0.59 \quad 0.43 \quad 0.34]$, $w = \frac{w_{RANK} * (1-\rho)}{\sum_i w_{RANK} * (1-\rho)} = [0.41 \quad 0.32 \quad 0.25]$
Step 7: $w_i^* = \frac{w_i / \sigma_i}{\sum_i w_i / \sigma_i} = [0.50 \quad 0.30 \quad 0.20]$

The final solution, $w^* = [0.50 \quad 0.30 \quad 0.20]$, portfolio risk $\sigma_p = \sqrt{w^{*/} * (\rho * \sigma * \sigma^/) * w^*} = 15.83$

Appendix: Data Sets

Data Set 1:
8 ETFs from the original "Forecast-Free Algorithms: A New Benchmark For Tactical Strategies" by David Varadi on August 9, 2011 post.
http://cssanalytics.wordpress.com/2011/08/09/forecast-free-algorithms-a-new-benchmark-for-tactical-strategies/

8 major asset classes/indices including:
1) S&P500
2) Nasdaq 100
3) Russell 2000
4) MSCI EAFE (Europe, Asia and Far-East)
5) Long-term Treasury Bonds
6) Real Estate and
7) Gold.
SPY,QQQ,EEM,IWM,EFA,TLT,IYR,GLD
Historical price data downloaded from Yahoo Finance

Data Set 2:
Free historical trade data for Futures and Forex data provided by Trading Blox from
http://www.tradingblox.com/tradingblox/free-historical-data.htm

This data set contains the Back-Adjusted Continuous Contracts for Futures and Cash for Forex. To compute returns, covariance matrix, and back-tests we re-created Futures returns using following formula:

$$\text{futures-return}[t] = \frac{(\text{unadjusted-futures}[t-1] + (\text{back-adjusted-futures}[t] - \text{back-adjusted-futures}[t-1]))}{\text{unadjusted-futures}[t-1]} - 1$$

Data Set 3:
Dow Stock (Engle)
AA,AXP,BA,CAT,DD,DIS,GE,IBM,IP,JNJ,JPM,KO,MCD,MMM,MO,MRK,MSFT
Historical price data downloaded from Yahoo Finance

Data Set 4:
Major Asset Classes (ETFs)
VTI,IEV,EEM,EWJ,AGG,GSG,GLD,ICF
Historical price data downloaded from Yahoo Finance

Data Set 5:
Current Nasdaq 100 Stocks (survivorship bias)
ATVI,ADBE,ALTR,AMZN,AMGN,APOL,AAPL,AMAT,ADSK,ADP,BBBY,BIIB,BMC,BRCM,CHRW,CA,CELG,CEPH,CERN,CHKP,CTAS,CSCO,CTXS,CTSH,CMCSA,COST,DELL,XRAY,DISH,EBAY,ERTS,EXPD,ESRX,FAST,FISV,FLEX,FLIR,FWLT,GILD,HSIC,HOLX,INFY,INTC,INTU,JBHT,KLAC,LRCX,LIFE,LLTC,LOGI,MAT,MXIM,MCHP,MSFT,MYL,NTAP,NWSA,NVDA,ORLY,ORCL,PCAR,PDCO,PAYX,PCLN,QGEN,QCOM,RIMM,ROST,SNDK,SIAL,SPLS,SBUX,SRCL,SYMC,TEVA,URBN,VRSN,VRTX,VOD,XLNX,YHOO
Historical price data downloaded from Yahoo Finance